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On Improved Generalized Versions of Bernstein's Inequality for Polynomial

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Abstract:

Let $v(z)$ ***be a polynomial of degree*** m ***having no zero zero in*** $|z| \le a$, $a \ge 1$, ***then for*** $1 \le T \le a$, ***Dewan and Bidkham [J. Math. Anal. Appl., vol. 166, pp. 319-324, 1992] proved***

$$\max_{|z|=T}|v'(z)| \le m\frac{(T+a)^{m-1}}{(1+a)^m}\max_{|z|=1}|v(z)|.$$

The result is best possible and extremal polynomial is $v(z)=(z+a)^m$.

In this paper, by considering a more general class of polynomials $v(z)$ having multiple zeros at the origin, we prove a result, which not only improves as well as generalizes the above result, but also has interesting consequences as special cases.

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1. INTRODUCTION

The famous Mathematician Bernstein [12, 14] investigated for the first time, an upper bound for the maximum modulus of the first derivative of a complex polynomial on the unit circle in terms the maximum modulus of the polynomial on the same circle and proved the following famous result known as Bernstein's inequality that if $v(z)$ is a polynomial of degree m, then

$$\max_{|z|=1}|v'(z)| \le m\max_{|z|=1}|v(z)|. \qquad (1.1)$$

Inequality (1.1) is best possible and equality occurs for $v(z)=\lambda z^m$, $\lambda \ne 0$, is any complex number.

If we restrict to the class of polynomials having no zero in $|z|<1$, then inequality (1.1) can be sharpened as

$$\max_{|z|=1}|v'(z)| \le \frac{m}{2}\max_{|z|=1}|v(z)|. \qquad (1.2)$$

The result is sharp and equality holds in (1.2) for $v(z)=\alpha+\beta z^m$, where $|\alpha|=|\beta|$.

Inequality (1.2) was conjectured by Erdös and later proved by Lax [10].

Simple proofs of this theorem were later given by de-Bruijn [4], and Aziz and Mohammad [1].

It was asked by Professor R.P. Boas that if $v(z)$ is a polynomial of degree m not vanishing in $|z|<a$, $a>0$, then how large can

$$\left\{ \max_{|z|=1}|v'(z)| \Big/ \max_{|z|=1}|v(z)| \right\} \text{ be ?} \qquad (1.3)$$

A partial answer to this problem was given by Malik [11], who proved

Theorem A. *If* $v(z)$ *is a polynomial of degree* m *having no zero in the disc* $|z|<a$, $a \ge 1$, *then*

$$\max_{|z|=1}|v'(z)| \le \frac{m}{1+a}\max_{|z|=1}|v(z)|. \qquad (1.4)$$

The result is best possible and equality holds for $v(z)=(z+a)^m$.

For the class of polynomials not vanishing in $|z|<a$, $a \le 1$, the precise estimate for maximum of $|v'(z)|$ on $|z|=1$, in general, does not seem to be easily obtainable.

For quite some time, it was believed that if $v(z)\neq 0$ in $|z|<a$, $a\leq 1$, then the inequality analogous to (1.4) should be

$$\max_{|z|=1}|v'(z)|\leq \frac{m}{1+a^m}\max_{|z|=1}|v(z)|. \quad (1.5)$$

till Professor E.B. Saff gave the example $v(z)=\left(z-\frac{1}{2}\right)\left(z+\frac{1}{3}\right)$ to counter this belief.

Dewan and Bidkham [6] generalized Theorem A by considering any circle that lies in a closed circular annulus of radii 1 and a where $a\geq 1$. In fact, they prove

Theorem B. *If* $v(z)$ *is a polynomial of degree* m *having no zero in* $|z|<a$, $a\geq 1$, *then for* $1\leq T\leq a$,

$$\max_{|z|=T}|v'(z)|\leq m\frac{(T+a)^{m-1}}{(1+a)^m}\max_{|z|=1}|v(z)|. \quad (1.6)$$

The result is best possible and extremal polynomial is $v(z)=(z+a)^m$.

In this paper, by considering a more general class of polynomials $v(z)$ having multiple zeros at the origin and also involving $\min_{|z|=a}|v(z)|$, we obtain the following two results where the first is a generalization and the second is an improvement and a generalization of (1.6) and maxima are considered on two different circles lying both inside and on any circle. More precisely, we prove

Theorem.1.1. *If* $v(z)=z^s\left(c_0+\sum_{\upsilon=\mu}^{m-s}c_\upsilon z^\upsilon\right)$, $1\leq\mu\leq m-s$, $0\leq s\leq m-1$, *is a polynomial of degree* m *with* s*-fold zero at the origin and the remaining* $m-s$ *zeros in* $|z|\geq a$, $a>0$, *then for* $0<t\leq T\leq a$,

$$\max_{|z|=T}|v'(z)|\leq\left[sT^{s-1}+(m-s)\frac{T^{\mu+s-1}}{T^\mu+a^\mu}\right]\times\left(\frac{T^\mu+a^\mu}{t^\mu+a^\mu}\right)^{\frac{m-s}{\mu}}\frac{1}{t^s}\max_{|z|=t}|v(z)|. \quad (1.7)$$

The result is best possible and equality in (1.7) *holds for* $v(z)=z^s\left(z^\mu+a^\mu\right)^{\frac{m-s}{\mu}}$, *where* $m-s$ *is a multiple of* μ.

Remark 1.1. Putting $s=0$, Theorem 1.1 gives the following result proved by Aziz and Shah [3], which further becomes a generalization of inequality (1.6) of Dewan and Bidkham [6].

Corollary 1.1. *If* $v(z)=c_0+\sum_{\upsilon=\mu}^{m}c_\upsilon z^\upsilon$, $1\leq\mu\leq m$, *is a polynomial of degree* m *having no zero in the disc* $|z|<a$, $a>0$, *then for* $0<t\leq T\leq a$,

$$\max_{|z|=T}|v'(z)|\leq\frac{mT^{\mu-1}\left(T^\mu+a^\mu\right)^{\frac{m}{\mu}-1}}{\left(t^\mu+a^\mu\right)^{\frac{m}{\mu}}}\max_{|z|=t}|v(z)|. \quad (1.9)$$

m The result is best possible and equality holds for the polynomial $v(z)=\left(z^\mu+a^\mu\right)^{\frac{m}{\mu}}$ *where* m *is a multiple of is* μ.

Remark 1.2. Putting $T=t=1$, Corollary 1.1 reduces to inequality (1.4) proved by Malik [11].

Next, under the same hypotheses, we prove an improvement of Theorem 1.1 by involving $\min_{|z|=a}|v(z)|$. In fact, we prove

Theorem.1.2. *If* $v(z)=z^s\left(c_0+\sum_{\upsilon=\mu}^{m-s}c_\upsilon z^\upsilon\right)$, $1\leq\mu\leq m-s$, $0\leq s\leq m-1$, *is a polynomial of degree* m *with* s*-fold zero at the origin and the remaining* $m-s$ *zeros no zero in* $|z|\geq a$, $a>0$, *then for* $0<t\leq T\leq a$,

$$\max_{|z|=T}|v'(z)|\leq sT^{s-1}\left[\left(\frac{T^\mu+a^\mu}{t^\mu+a^\mu}\right)^{\frac{m-s}{\mu}}\times\left\{\frac{1}{t^s}\max_{|z|=t}|v(z)|-\frac{1}{a^s}\min_{|z|=a}|v(z)|\right\}+\frac{1}{a^s}\min_{|z|=a}|v(z)|\right]+(m-s)\times\frac{T^{\mu+s-1}}{T^\mu+a^\mu}\left[\left(\frac{T^\mu+a^\mu}{t^\mu+a^\mu}\right)^{\frac{m-s}{\mu}}\left\{\frac{1}{t^s}\max_{|z|=t}|v(z)|-\frac{1}{a^s}\min_{|z|=a}|v(z)|\right\}\right]. \quad (1.8)$$

As in Theorem 1.1, *equality in* (1.8) *holds for*

$v(z)=z^s\left(z^\mu+a^\mu\right)^{\frac{m-s}{\mu}}$, *where* $m-s$ *is a multiple of* μ.

Remark 1.3. Putting $s=0$, Theorem 1.2 gives the following result proved by Aziz and Shah [3] and it improves Corollary 1.1.

Corollary 1.2. *If* $v(z)=c_0+\sum_{\upsilon=\mu}^{m}c_\upsilon z^\upsilon$, $1\le\mu\le m$, *is a polynomial of degree* m *having no zero in* $|z|<a$, $a>0$, *then for* $0<t\le T\le a$,

$$\max_{|z|=T}|v'(z)|\le\frac{mT^{\mu-1}\left(T^\mu+a^\mu\right)^{\frac{m}{\mu}-1}}{\left(t^\mu+a^\mu\right)^{\frac{m}{\mu}}}\times\left\{\max_{|z|=t}|v(z)|-\min_{|z|=a}|v(z)|\right\}. \tag{1.9}$$

The result is best possible for the polynomial $v(z)=\left(z^\mu+a^\mu\right)^{\frac{m}{\mu}}$ *where* m *is a multiple of* μ.

Remark 1.4. If we put $\mu=t=1$ in Corollary 1.2, it gives an improved bound of (1.6) proved by Dewan and Bidkham [6].

Remark 1.5. Putting $T=t=1$ Theorem 1.2 again provides a generalization of a result proved by Pukhta [13].

Corollary 1.3. *If* $v(z)=z^s\left(c_0+\sum_{\upsilon=\mu}^{m-s}c_\upsilon z^\upsilon\right)$,

$1\le\mu\le m-s$, $0\le s\le m-1$, *is a polynomial of degree n with* s*-fold zero at the origin and the remaining* $m-s$ *zeros in* $|z|\ge a$, $a>0$, *then*

$$\max_{|z|=1}|v'(z)|\le s\max_{|z|=1}|v(z)|+(m-s)\frac{1}{1+a^\mu}\left\{\max_{|z|=1}|v(z)|-\min_{|z|=a}|v(z)|\right\}. \tag{1.10}$$

Equality in (1.10) *holds for* $v(z)=z^s\left(z^\mu+a^\mu\right)^{\frac{m-s}{\mu}}$, *where* $m-s$ *is a multiple of* μ.

Remark 1.6. If we assign $s=0$ and $\mu=1$ in Corollary 1.3, it reduces to a result of Govil [8], which again improves Malik's well-known inequality (1.4).

Remark 1.7. If we put $s=0$ and $\mu=a=1$, Corollary 1.3 further gives an improved version of inequality (1.2), conjectured by Erdös and later proved by Lax [10].

2. LEMMA

The following lemmas are needed for the proof of the theorem.

Lemma 2.1. *If* $v(z)=c_0+\sum_{\upsilon=\mu}^{m}c_\upsilon z^\upsilon$, $1\le\mu\le m$, *is a polynomial of degree* m *such that* $v(z)\ne0$ *in* $|z|<a$, $a>0$, *then*

$$\max_{|z|=1}|v'(z)|\le\frac{m}{1+a^\mu}\max_{|z|=1}|v(z)|. \tag{2.1}$$

The result is best possible and equality holds for *for* $v(z)=\left(z^\mu+a^\mu\right)^{\frac{m}{\mu}}$, *where* m *is a multiple of* μ.

The above result is due to Chan and Malik [5].

Lemma 2.2. *If* $v(z)=c_0+\sum_{\upsilon=\mu}^{m}c_\upsilon z^\upsilon$, $1\le\mu\le m$, *is a polynomial of degree* m *such that* $v(z)\ne0$ *in* $|z|<a$, $a>0$, *then for* $0<t\le T\le a$,

$$\max_{|z|=T}|v(z)|\le\left(\frac{T^\mu+a^\mu}{t^\mu+a^\mu}\right)^{\frac{m}{\mu}}\max_{|z|=t}|v(z)|. \tag{2.2}$$

Equality holds in (2.2) *for* $v(z)=\left(z^\mu+a^\mu\right)^{\frac{m}{\mu}}$ *where* m *is a multiple of* μ.

This Lemma is due to Jain [9].

Lemma 2.3. *If* $v(z)=c_0+\sum_{\upsilon=\mu}^{m}c_\upsilon z^\upsilon$, $1\le\mu\le m$, *is a polynomial of degree* m *such that* $v(z)\ne0$ *in* $|z|<a$, $a>0$, *then for* $0<t\le T\le a$,

$$\max_{|z|=T}|v(z)|\le A\max_{|z|=t}|v(z)|-(A-1)\min_{|z|=a}|v(z)|. \tag{2.3}$$

where

$$A=\left(\frac{T^\mu+a^\mu}{t^\mu+a^\mu}\right)^{\frac{m}{\mu}}.$$

Equality occurs in (2.2) *for* $v(z)=\left(z^\mu+a^\mu\right)^{\frac{m}{\mu}}$ *where* m *is a multiple of* μ.

This Lemma was proved by Dewan et. al [7].

3. PROOF OF THE THEOREM

Proof of Theorem 1.2. Let $v(z)=z^s h(z)$, (3.1)

where $h(z)=c_0+\sum_{v=\mu}^{m-s} c_v z^v$ is a polynomial of degree $m-s$ having no zero in $|z|<a$, m. Then for $T\le a$, $H(z)=h(Tz)$ has no zero in $|z|<\frac{a}{T}$, $\frac{a}{T}\ge 1$.

Now $$\min_{|z|=\frac{a}{T}}|H(z)|=\min_{|z|=\frac{a}{T}}|h(Tz)|=\min_{|z|=a}|h(z)|=k.$$

By Rouche's theorem, for every real or complex number λ, with $|\lambda|<1$. The polynomial $H(z)-\lambda k$ is of degree $m-s$ and has no zero in $|z|<\frac{a}{T}$, $\frac{a}{T}\ge 1$. Hence, on applying Lemma 2.1 to $H(z)-\lambda k$, we have

$$T\max_{|z|=1}|h'(Tz)|\le (m-s)\frac{1}{1+\left(\frac{a}{T}\right)^{\mu}}\max_{|z|=1}|h(Tz)-\lambda k|,$$

which is equivalent to

$$\max_{|z|=T}|h'(z)|\le (m-s)\frac{T^{\mu-1}}{T^{\mu}+a^{\mu}}\max_{|z|=a}|h(z)-\lambda k|. \qquad (3.2)$$

Let z_0 on $|z|=T$ be such that

$$\max_{|z|=T}|h(z)-\lambda k|=|h(z_0)-\lambda k|. \qquad (3.3)$$

Further, we choose the argument of λ in the right hand side of (3.3) such that

$$|h(z_0)-\lambda k|=|h(z_0)|-|\lambda|k$$
$$\le \max_{|z|=T}|h(z)|-|\lambda|k. \qquad (3.4)$$

Using (3.4) in (3.2), we get

$$\max_{|z|=T}|h'(z)|\le (m-s)\frac{T^{\mu-1}}{T^{\mu}+a^{\mu}}\left\{\max_{|z|=T}|h(z)|-|\lambda|k\right\}. \qquad (3.5)$$

Since $h(z)$ has no zero in $|z|<a$, $a>0$, applying Lemma 2.3 to $h(z)$, we have for $0<t\le T\le a$,

$$\max_{|z|=T}|h(z)|\le\left(\frac{T^{\mu}+a^{\mu}}{t^{\mu}+a^{\mu}}\right)^{\frac{m-s}{\mu}}\max_{|z|=t}|h(z)| -\left(\left(\frac{T^{\mu}+a^{\mu}}{t^{\mu}+a^{\mu}}\right)^{\frac{m-s}{\mu}}-1\right)\min_{|z|=a}|h(z)|, \qquad (3.6)$$

If we use (3.6) in (3.5), we obtain

$$\max_{|z|=T}|h'(z)|\le (m-s)\frac{T^{\mu-1}}{T^{\mu}+a^{\mu}}\left[\left(\frac{T^{\mu}+a^{\mu}}{t^{\mu}+a^{\mu}}\right)^{\frac{m-s}{\mu}}\times\left\{\max_{|z|=t}|h(z)|-\min_{|z|=a}|h(z)|\right\}+\min_{|z|=a}|h(z)|-|\lambda|k\right]. \qquad (3.7)$$

From (3.1), we have

$$zv'(z)=sz^s h(z)+z^{s+1}h'(z)=sv(z)+z^{s+1}h'(z),$$

from which, we have on $|z|=T$

$$T|v'(z)|\le s|v(z)|+T^{s+1}|h'(z)|,$$

That is for $|z|=T$

$$|v'(z)|\le\frac{s}{T}|v(z)|+T^s\max_{|z|=T}|h'(z)|. \qquad (3.8)$$

Combining (3.7) and (3.8), we have for $|z|=T$

$$|v'(z)|\le\frac{s}{T}|v(z)|+(m-s)\frac{T^{s+\mu-1}}{T^{\mu}+a^{\mu}}\left[\left(\frac{T^{\mu}+a^{\mu}}{t^{\mu}+a^{\mu}}\right)^{\frac{m-s}{\mu}}\times\left\{\max_{|z|=t}|h(z)|-\min_{|z|=a}|h(z)|\right\}+\min_{|z|=a}|h(z)|-|\lambda|k\right]. \qquad (3.8)$$

Again, from (3.1) $v(z)=z^s h(z)$ and thus we have the relations

$$\max_{|z|=T}|v(z)|=T^s\max_{|z|=T}|h(z)|.$$

and

$$k=\min_{|z|=a}|h(z)|=\frac{1}{a^s}\min_{|z|=a}|v(z)|.$$

Making use of these two relations in (3.8) and letting limit as $|\lambda|\to 1$, we have for $0<t\le T\le a$,

$$\max_{|z|=T}|v'(z)| \le sT^{s-1}\left[\left(\frac{T^{\mu}+a^{\mu}}{t^{\mu}+a^{\mu}}\right)^{\frac{m-s}{\mu}}\right.$$
$$\times\left\{\frac{1}{t^s}\max_{|z|=r}|v(z)|-\frac{1}{a^s}\min_{|z|=a}|v(z)|\right\}+\left.\frac{1}{a^s}\min_{|z|=a}|v(z)|\right]$$
$$+(m-s)\frac{T^{s+\mu-1}}{T^{\mu}+a^{\mu}}\left[\left(\frac{T^{\mu}+a^{\mu}}{t^{\mu}+a^{\mu}}\right)^{\frac{m-s}{\mu}}\right.$$
$$\left.\times\left\{\frac{1}{t^s}\max_{|z|=t}|v(z)|-\frac{1}{a^s}\min_{|z|=a}|v(z)|\right\}\right],$$

this completes the proof of the Theorem 1.2.

Proof of Theorem 1.1. The proof of this theorem follows on the similar lines as that of Theorem 1.2. For the sake of completeness, we give some hints:

Let $v(z)=z^s h(z)$, where $h(z)=c_0+\sum_{\upsilon=\mu}^{m-s}c_{\upsilon}z^{\upsilon}$ is a polynomial of degree $m-s$ having no zero in $|z|<a$, $a>0$. Then for $T\le a$, $H(z)=h(Tz)$ is of degree $m-s$ and has no zero in $|z|<\frac{a}{T}$, $\frac{a}{T}\ge 1$.

The main thing is to apply Lemma 2.1 to $H(z)$ instead of to $H(z)-\lambda k$ in the proof of Theorem 1.2 and application of Lemma 2.2 instead of Lemma 2.3 is followed.

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